

GAP DISTRIBUTION OF FAREY FRACTIONS DETERMINED BY SUBGROUPS OF $SL(2, \mathbb{Z})$

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ABSTRACT. For a given finite index subgroup $H \subseteq SL(2, \mathbb{Z})$, we use a process developed by Fisher and Schmidt [12] to lift a Poincaré section of the horocycle flow on $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ found by Athreya and Cheung [3] to the finite cover $SL(2, \mathbb{R})/H$ of $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$. We then use the properties of this section to prove the existence of the limiting gap distribution of various subsets of Farey fractions.

1. INTRODUCTION

The significant work of Elkies and McMullen [10] and of Marklof and Strömbergsson [18] has demonstrated that ergodic properties of homogeneous flows can provide a very powerful device in the study of the limiting gap distributions of certain sequences of arithmetic origin. Recently, Athreya and Cheung [3] realized the horocycle flow on $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ as a suspension flow over the BCZ map introduced by Boca, Cobeli, and Zaharescu [7] in their study of statistical properties of Farey fractions. Athreya and Cheung used this connection to rederive the limiting gap distribution and other properties of Farey fractions. The process used in [3] to obtain these results was later generalized in [1] to explain the gap distributions of various different sequences. Also, recent work of Fisher and Schmidt [12] was concerned with lifting Poincaré sections of the geodesic flow on the unit tangent bundle of the modular surface to a finite cover, with the primary aim of obtaining statistical properties of continued fractions.

In this paper, we use the process in [12] to explicitly lift, for every finite index subgroup $H \subseteq SL(2, \mathbb{Z})$, the section of the horocycle flow discovered in [3] to the cover $SL(2, \mathbb{R})/H$ of $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$. As an application, we establish the existence of the limiting gap measure of certain subsets of Farey fractions, following the ideas in [3] and [1]. A given subset we consider is determined, as described below, by a finite index subgroup $H \subseteq SL(2, \mathbb{Z})$, and corresponds to a Poincaré section of $SL(2, \mathbb{Z})/H$ obtained by intersecting the lift of the section in [3] with certain sheets of the cover $SL(2, \mathbb{R})/H \rightarrow SL(2, \mathbb{R})/SL(2, \mathbb{Z})$.

Recall that the Farey sequence of order Q is the set $\mathcal{F}(Q)$ of fractions $\frac{a}{q} \in [0, 1]$ such that $(a, q) = 1$ and $q \leq Q$. Various properties regarding the spacing statistics of the increasing sequence $(\mathcal{F}(Q))$ of subsets of $[0, 1]$ have been studied [13], [4], [9]. Additionally, certain subsets of Farey sequences have been considered. For instance, if $\mathcal{F}_{Q,d} \subseteq \mathcal{F}(Q)$ is the set of fractions $\frac{a}{q}$ with $(q, d) = 1$ and $\tilde{\mathcal{F}}_{Q,\ell} \subseteq \mathcal{F}(Q)$ is the set of fractions $\frac{a}{q}$ with $\ell \nmid a$, then the number of pairs $(\frac{a}{q}, \frac{a'}{q'})$ of consecutive fractions in $\mathcal{F}_{Q,d}$ with fixed $a'q - aq' = k$ has been estimated by Badziahin and Haynes [5], the pair correlation function of the sequence (\mathcal{F}_{Q,d_Q}) was shown to exist by Xiong and Zaharescu [20] where d_Q varies with Q subject to the constraints $d_{Q_1} \mid d_{Q_2}$ as $Q_1 < Q_2$ and $d_Q \ll Q^{\log \log Q/4}$, and

the limiting gap distribution measure for the sequences $(\mathcal{F}_{Q,d})$ and $(\tilde{\mathcal{F}}_{Q,\ell})$ were shown to exist for fixed d and ℓ by Boca, Spiegelhalter, and the author [8].

For a given finite subset $A = \{x_0 \leq x_1 \leq \dots \leq x_N\}$ of $[0, 1]$, we define the *gap distribution measure* of A to be the probability measure ν_A on $[0, \infty)$ such that

$$\nu_A[0, \xi] = \frac{1}{N} \#\{j \in [1, N] : N(x_j - x_{j-1}) \leq \xi(x_N - x_0)\}, \quad \xi \geq 0.$$

For a sequence (A_n) of finite subsets of $[0, 1]$, we call the weak limit of (ν_{A_n}) , if it exists, the *limiting gap measure* of (A_n) .

Let $G = \text{SL}(2, \mathbb{R})$ and $\Gamma = \text{SL}(2, \mathbb{Z})$. We prove the following result:

Theorem 1. *Let H be a finite index subgroup of Γ and $M \subseteq \Gamma/H$ be a nonempty subset, closed under left multiplication by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Also, for $Q \in \mathbb{N}$, let $\mathcal{F}_M(Q) \subseteq \mathcal{F}(Q)$ be the set of fractions $\frac{a}{q}$ such that*

$$\begin{pmatrix} q' & a' \\ -q & -a \end{pmatrix} H \in M,$$

where $\frac{a'}{q'}$ is the successor of $\frac{a}{q}$ in $\mathcal{F}(Q)$. Then the sequence $(\mathcal{F}_M(Q))$ becomes equidistributed in $[0, 1]$ as $Q \rightarrow \infty$. Furthermore, if $I \subseteq [0, 1]$ is a given subinterval and $\mathcal{F}_{I,M}(Q) = \mathcal{F}_M(Q) \cap I$, then the limiting gap measure $\nu_{I,M}$ of $(\mathcal{F}_{I,M}(Q))$ exists and has a continuous and piecewise real-analytic density.

We include the hypothesis that M is closed under left multiplication by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ in order to ensure that $(\mathcal{F}_M(Q))$ is an increasing sequence of sets. Indeed, let $\frac{a}{q} \in \mathcal{F}_M(Q)$ so that

$$\begin{pmatrix} q' & a' \\ -q & -a \end{pmatrix} H \in M,$$

where $\frac{a'}{q'}$ is the successor of $\frac{a}{q}$ in $\mathcal{F}(Q)$. If $Q' \geq Q$, then by the mediant property of Farey fractions, the successor of $\frac{a}{q}$ in $\mathcal{F}(Q')$ is equal to $\frac{na+a'}{nq+q'}$ for some $n \geq 0$. We then have

$$\begin{pmatrix} nq+q' & na+a' \\ -q & -a \end{pmatrix} H = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} q' & a' \\ -q & -a \end{pmatrix} H \in M,$$

implying that $\frac{a}{q} \in \mathcal{F}_M(Q')$, and hence $(\mathcal{F}_M(Q))$ is increasing.

Applying Theorem 1 with $H = \Gamma(m)$, where m is a positive integer and $\Gamma(m)$ is the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{m} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

of Γ , and with

$$M = \left\{ \begin{pmatrix} n_4 & n_3 \\ -n_2 & -n_1 \end{pmatrix} H \in \Gamma/H : (n_1, n_2) \pmod{m} \in A \right\}$$

where $A \subseteq (\mathbb{Z}/m\mathbb{Z})^2$ is such that M is nonempty, i.e., there is some $(n_1, n_2) \in A$ such that $(n_1, n_2, m) = 1$, we have the following result:

Corollary 1. *Let $A \subseteq (\mathbb{Z}/m\mathbb{Z})^2$ be such that $(n_1, n_2, m) = 1$ for some $(n_1, n_2) \in A$, and let $I \subseteq [0, 1]$ be a subinterval. Then for $Q \in \mathbb{N}$, let $\mathcal{F}_{m,A}(Q)$ be the set of fractions $\frac{a}{q} \in \mathcal{F}(Q)$ such that*

$(a, q) \equiv (n_1, n_2) \pmod m$ for some $(n_1, n_2) \in A$ and let $\mathcal{F}_{I,m,A}(Q) = \mathcal{F}_{m,A}(Q) \cap I$. Then $(\mathcal{F}_{m,A}(Q))$ becomes equidistributed in $[0, 1]$ as $Q \rightarrow \infty$, and the limiting gap measure of $(\mathcal{F}_{I,m,A}(Q))$ exists and has a continuous and piecewise real-analytic density.

This corollary includes the existence of the limiting gap measures of $(\mathcal{F}_{Q,d})$ and $(\tilde{\mathcal{F}}_{Q,\ell})$ proven in [8] as special cases since $\mathcal{F}_{Q,d} = \mathcal{F}_{d,A}(Q)$, where $A \subseteq (\mathbb{Z}/m\mathbb{Z})^2$ is the subset consisting of all pairs (n_1, n_2) such that $(n_2, d) = 1$, and $\tilde{\mathcal{F}}_{Q,\ell} = \mathcal{F}_{\ell,A'}$, where $A' \subseteq (\mathbb{Z}/m\mathbb{Z})^2$ is the subset having all pairs (n_1, n_2) with $n_1 \not\equiv 0 \pmod \ell$. Congruence subgroups also appear in the study [18, Corollary 2.7] of the related problem of proving the existence of the limiting gap measure for the angles of visible points in \mathbb{Z}^2 with respect to an observer at a rational point. See [2] for a similar application of the ergodic properties of the horocycle flow on G/H , in the case where H is the Hecke $(2, 5, \infty)$ triangle group, to the computation of the limiting gap measure of slopes on the golden L.

In Section 2, we review the work of Athreya and Cheung [3] in showing the horocycle flow as a suspension flow by giving a Poincaré section Ω' of the horocycle flow in which the first return map is the BCZ map. We also outline how Ω' relates to Farey fraction gaps and how Athreya and Cheung used the equidistribution properties of the horocycle flow to prove results about those gaps. Then in Sections 3–6, we prove Theorem 1 using the same process. We start in Section 3 by proving that $\bigcup_{Q \in \mathbb{N}} \mathcal{F}_M(Q)$ is dense in $[0, 1]$, and this involves proving an elementary lemma regarding representatives for cosets in Γ/H . In Section 4, we use results in [12] to construct a Poincaré section of the horocycle flow on G/H analogous to Ω' that relates to the gaps in $(\mathcal{F}_M(Q))$. In Section 5, we prove some important properties of the first return time function of the Poincaré section which have an effect on the existence and properties of $\nu_{I,M}$ mentioned in Theorem 1. Then in Section 6, we prove the weak convergence of a certain sequence of measures on the Poincaré section, analogous to the sequence $(\rho_{Q,I})$ in [3] which we mention in Section 2. From this convergence we can conclude the existence of the limiting gap measure of $(\mathcal{F}_{I,M}(Q))$. Lastly in Section 7, we examine a particular property, which we call the repulsion gap, of $\nu_{I,M}$ in some particular cases.

2. THE BCZ MAP AND THE POINCARÉ SECTION OF ATHREYA AND CHEUNG

Let $\mathcal{T} \subseteq \mathbb{R}^2$ be the Farey triangle containing the points (a, b) satisfying $0 < a, b \leq 1$ and $a + b > 1$. The BCZ map $T : \mathcal{T} \rightarrow \mathcal{T}$ is defined in [7] by

$$T(a, b) = \left(b, \left\lfloor \frac{1+a}{b} \right\rfloor b - a \right).$$

This map has relevance to Farey fractions as follows: if $\frac{a}{q} < \frac{a'}{q'} < \frac{a''}{q''}$ are consecutive fractions in $\mathcal{F}(Q)$, then $a'' = Ka' - a$ and $q'' = Kq' - q$, where $K = \lfloor \frac{Q+q}{q'} \rfloor = \lfloor \frac{1+q/Q}{q'/Q} \rfloor$. As a result, $\frac{q''}{Q} = K\frac{q'}{Q} - \frac{q}{Q}$, which is the second coordinate of $T(\frac{q}{Q}, \frac{q'}{Q})$. Hence we have the equality

$$T\left(\frac{q}{Q}, \frac{q'}{Q}\right) = \left(\frac{q'}{Q}, \frac{q''}{Q}\right).$$

So if we consider the correspondence $\frac{a}{q} \leftrightarrow (\frac{q}{Q}, \frac{q'}{Q})$ of $\mathcal{F}(Q)$ with certain points of \mathcal{T} , we see that \mathcal{T} in some sense parameterizes $\mathcal{F}(Q)$, and T maps a point corresponding to a Farey fraction to the point corresponding to the succeeding fraction.

In [3], the Farey triangle was viewed as a subset of G/Γ by letting

$$P = \left\{ p_{a,b} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : (a,b) \in \mathcal{T} \right\}$$

and considering the set

$$\Omega' = P\Gamma/\Gamma = \{\Lambda_{a,b} = p_{a,b}\Gamma : (a,b) \in \mathcal{T}\},$$

which was found to be a Poincaré section for the horocycle flow, viewed as the action by left multiplication on G/Γ of

$$N = \left\{ h_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} : s \in \mathbb{R} \right\}.$$

This means that if we let $\mu_{G/\Gamma}$ be the Haar measure on G/Γ , normalized so that $\mu_{G/\Gamma}(G/\Gamma) = \frac{\pi^2}{3}$, then for $\mu_{G/\Gamma}$ -a.e. $\Lambda \in G/\Gamma$, the set $\{s \in \mathbb{R} : h_s\Lambda \in \Omega'\}$ of times the orbit of Λ under the horocycle flow meets Ω' is nonempty, countable, and discrete. The first return time function $r : \Omega' \rightarrow \mathbb{R}$ defined by $r(\Lambda_{a,b}) = \min\{s > 0 : h_s\Lambda_{a,b} \in \Omega'\}$ is $r(\Lambda_{a,b}) = \frac{1}{ab}$, and the first return map $R : \Omega' \rightarrow \Omega'$ defined by $R(\Lambda_{a,b}) = h_{r(\Lambda_{a,b})}\Lambda_{a,b}$ is $R(\Lambda_{a,b}) = \Lambda_{T(a,b)}$, where T is the BCZ map. The last equality can be seen by the calculation

$$\begin{aligned} h_{r(\Lambda_{a,b})}\Lambda_{a,b} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{ab} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \Gamma = \begin{pmatrix} a & b \\ -b^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \lfloor \frac{1+a}{b} \rfloor \end{pmatrix} \Gamma \\ &= \begin{pmatrix} b & \lfloor \frac{1+a}{b} \rfloor b - a \\ 0 & b^{-1} \end{pmatrix} \Gamma = \Lambda_{T(a,b)}. \end{aligned}$$

Also, if we identify G/Γ with the set $\{(a,b,s) : (a,b) \in \mathcal{T}, 0 \leq s < \frac{1}{ab}\}$ via the correspondence $h_s\Lambda_{a,b} \leftrightarrow (a,b,s)$, then $d\mu_{G/\Gamma} = 2 da db ds$.

Now let

$$\mathcal{F}(Q) = \left\{ \gamma_0 = \frac{a_0}{q_0} = \frac{0}{1} < \gamma_1 = \frac{a_1}{q_1} < \dots < \gamma_{N(Q)} = \frac{a_{N(Q)}}{q_{N(Q)}} = \frac{1}{1} \right\}.$$

Then letting $I \subseteq [0, 1]$ be a subinterval, $\mathcal{F}_I(Q) = \mathcal{F}(Q) \cap I$, and $N_I(Q) = \#\mathcal{F}_I(Q)$, we define on Ω' the measure

$$\rho_{Q,I} = \frac{1}{N_I(Q)} \sum_{i:\gamma_i \in I} \delta_{R^i(\Lambda_{1,1/Q})} = \frac{1}{N_I(Q)} \sum_{i:\gamma_i \in I} \delta_{\Lambda_{q_i/Q, q_{i+1}/Q}}.$$

Notice that $r(\Lambda_{q_i/Q, q_{i+1}/Q}) = \frac{Q^2}{q_i q_{i+1}} = Q^2(\gamma_{i+1} - \gamma_i)$, and as a result,

$$\frac{\#\{\gamma_i \in \mathcal{F}_I(Q) : \frac{3}{\pi^2} Q^2(\gamma_{i+1} - \gamma_i) \in [0, c]\}}{N_I(Q)} = \rho_{Q,I} \left(r^{-1} \left[0, \frac{\pi^2}{3} c \right] \right) \quad (2.1)$$

for all $c \geq 0$. Now $N_I(Q) \sim \frac{3}{\pi^2} |I| Q^2$, and if $\gamma_{I,l}$ and $\gamma_{I,g}$ are the least and greatest elements in $\mathcal{F}_I(Q)$, respectively (we suppress the dependence on Q), then $\gamma_{I,g} - \gamma_{I,l} \rightarrow |I|$ as $Q \rightarrow \infty$. So the limit of the left side of (2.1) as $Q \rightarrow \infty$ is the measure of $[0, c]$ under the limiting gap measure of $(\mathcal{F}_I(Q))$. So to show that the limiting gap measure of $(\mathcal{F}_I(Q))$ exists, it suffices to prove that the right side of (2.1) exists. To do so, Athreya and Cheung proved that the sequence $(\rho_{Q,I})$ of measures converges in the weak-* topology to the measure m on Ω' given by $dm = 2 da db$. They first noticed that if $\rho_{Q,I}^R$ is the measure on G/Γ defined by $d\rho_{Q,I}^R = d\rho_{Q,I} ds$, where we are viewing

G/Γ as the set $\{(\Lambda_{a,b}, s) \in \Omega' \times \mathbb{R} : 0 \leq s < \frac{1}{ab}\}$ by the correspondence $h_s \Lambda_{a,b} \leftrightarrow (\Lambda_{a,b}, s)$, then $\rho_{Q,I}^R \rightarrow \mu_{G/\Gamma}$ in the weak-* topology. This convergence is a consequence of the equidistribution of closed horocycles in G/Γ , proven originally by Sarnak [19] and reproven by Eskin and McMullen [11].

It then follows that if $\pi_{\Omega'} : G/\Gamma \rightarrow \Omega'$ is the projection $(\Lambda_{a,b}, s) \mapsto \Lambda_{a,b}$ (we are again viewing G/Γ as $\{(\Lambda_{a,b}, s) \in \Omega' \times \mathbb{R} : 0 \leq s < \frac{1}{ab}\}$), then

$$\frac{1}{r} \pi_{\Omega'}^* \rho_{Q,I}^R \rightarrow \frac{1}{r} \pi_{\Omega'}^* \mu_{G/\Gamma} \quad (\text{as } Q \rightarrow \infty)$$

in the weak-* topology. It is easy to see that $\rho_{Q,I} = \frac{1}{r} \pi_{\Omega'}^* \rho_{Q,I}^R$ and $m = \frac{1}{r} \pi_{\Omega'}^* \mu_{G/\Gamma}$, and so $\rho_{Q,I} \rightarrow m$.

Remark 1. The convergence $\rho_{Q,I} \rightarrow m$ was proven in [15] in the case $I = [0, 1]$. This convergence also follows from [16, Theorem 6], in which the equidistribution of Farey points of arbitrary dimension was proven. See [17] for results regarding the spacing statistics of higher-dimensional Farey fractions. Using $\rho_{Q,I} \rightarrow m$, Athreya and Cheung [3] not only explained the gap distribution of $(\mathcal{F}_I(Q))$, but through finding appropriate functions $F : \Omega' \rightarrow \mathbb{R}$ such that

$$\lim_{Q \rightarrow \infty} \int F d\rho_{Q,I} = \int F dm,$$

they were able to recast in this unified setting many previously known results about Farey fractions, including results on their h -spacings and indices.

3. THE DENSITY OF $\bigcup_{Q \in \mathbb{N}} \mathcal{F}_M(Q)$ IN $[0, 1]$

Throughout this section and Sections 4–6, let $H \subseteq \Gamma$ be a subgroup of finite index, $M = \{m_1 H, \dots, m_k H\} \subseteq \Gamma/H$ be a nonempty subset closed under left multiplication by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and $I = [x_1, x_2] \subseteq [0, 1]$ be a subinterval. We now set out to prove that the limiting gap measure $\nu_{I,M}$ of $(\mathcal{F}_{I,M}(Q))$ exists and start in this section by proving that $\bigcup_{Q \in \mathbb{N}} \mathcal{F}_M(Q)$ is dense in $[0, 1]$. We first prove the following elementary lemma:

Lemma 1. *Let gH be any coset in Γ/H . Then there exist positive integers a, b, c, d such that*

$$\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \in gH.$$

Proof. First note that since $[\Gamma : H] < \infty$, there exists an integer $P \geq 2$ such that

$$U_P = \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix}, \quad L_P = \begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix} \in H.$$

Let $A = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in gH$. If $a_0 > 0$ and $b_0 = 0$, then $A = \begin{pmatrix} 1 & 0 \\ c_0 & 1 \end{pmatrix}$. Replacing A by AL_P^{-j} for a large enough j replaces c_0 by a number less than -1 , and so we can assume that $c_0 < -1$. We then have $AU_P = \begin{pmatrix} 1 & P \\ c_0 & c_0 P + 1 \end{pmatrix}$, which is a matrix of the desired form. So the proof is complete in this case.

So assume that $a_0 \leq 0$ or $b_0 \neq 0$. If $b_0 = 0$, we must have $a_0 < 0$, and if $a_0 = 0$ so that $b_0 \neq 0$, multiplying A on the right by L_P or L_P^{-1} replaces a_0 by a negative number. So we can assume that $a_0 < 0$. Then multiplying A on the right by U_P^j for a large enough j replaces b_0

by a positive number, and so assume that $b_0 > 0$. Now since $a_0 d_0 - b_0 c_0 = 1$, we clearly have $c_0 d_0 \leq 0$. Suppose that $d_0 = 0$, implying that $A = \begin{pmatrix} a_0 & 1 \\ -1 & 0 \end{pmatrix}$. Multiplying A on the right by $L_P^j U_P$ yields $\begin{pmatrix} a_0 + jP & P(a_0 + jP) + 1 \\ -1 & -P \end{pmatrix}$. Choosing j so that $a_0 + jP > 0$ yields a matrix of the desired form, and so the proof is complete in this case. If $c_0 = 0$, then $A = \begin{pmatrix} -1 & b_0 \\ 0 & -1 \end{pmatrix}$, and multiplying A on the right by L_P^{-1} yields $\begin{pmatrix} -1 - b_0 P & b_0 \\ P & -1 \end{pmatrix}$.

Thus we have reduced the case where $c_0 = 0$ to the situation where $A = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$ with $a, b, c, d > 0$, which we now consider. Let $\frac{\gamma}{\alpha} < \frac{\delta}{\beta}$ be fractions such that $\alpha\delta - \beta\gamma = 1$ and $\frac{a}{b} < \frac{\gamma}{\alpha}$. Matrix multiplication reveals that any power of $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is of the form $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ where $\frac{\gamma}{\alpha} \leq \frac{\gamma'}{\alpha'} < \frac{\delta'}{\beta'} \leq \frac{\delta}{\beta}$. So noting that $[\Gamma : H] < \infty$, we can replace B by some power of B that is in H . We then have $AB \in gH$ and

$$AB = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -a\alpha + b\gamma & -a\beta + b\delta \\ c\alpha - d\gamma & c\beta - d\delta \end{pmatrix},$$

which is a matrix of the desired form since $\frac{c}{d} < \frac{a}{b} < \frac{\gamma}{\alpha} < \frac{\delta}{\beta}$.

The last case we need to consider is when $A = \begin{pmatrix} -a & b \\ -c & d \end{pmatrix}$ with $a, b, c, d > 0$. Noting that $\frac{a}{b} < \frac{c}{d}$ since $-ad + bc = 1$, we can find fractions $\frac{\gamma}{\alpha} < \frac{\delta}{\beta}$ such that $\alpha\delta - \beta\gamma = 1$ and $\frac{a}{b} < \frac{\gamma}{\alpha} < \frac{\delta}{\beta} < \frac{c}{d}$. As in the previous case, we may assume that $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H$. Then $AB \in gH$ and

$$AB = \begin{pmatrix} -a & b \\ -c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -a\alpha + b\gamma & -a\beta + b\delta \\ -c\alpha + d\gamma & -c\beta + d\delta \end{pmatrix}$$

is a matrix of the desired form. This completes the proof. \square

So by Lemma 1, there exist $a, b, c, d \in \mathbb{N}$ such that $\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} H \in M$. Multiplying $\begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ on the right by L_P yields a matrix of the form $\begin{pmatrix} q' & a' \\ -q & -a \end{pmatrix}$, where $\frac{a}{q} < \frac{a'}{q'}$ are consecutive Farey fractions of some order. We have $\begin{pmatrix} q' & a' \\ -q & -a \end{pmatrix} H \in M$, proving that $\bigcup_{Q=1}^{\infty} \mathcal{F}_M(Q)$ is nonempty.

Lemma 2. *The set $\bigcup_{Q=1}^{\infty} \mathcal{F}_M(Q)$ is dense in $[0, 1]$.*

Proof. Let $I' \subseteq [0, 1]$ be any open subinterval and let $\frac{a}{q} < \frac{a'}{q'}$ be consecutive Farey fractions of some order in I' . As noted in Lemma 1, any power of $A = \begin{pmatrix} q & a \\ q' & a' \end{pmatrix}$ is of the form $\begin{pmatrix} v & u \\ v' & u' \end{pmatrix}$, where $\frac{u}{v}$ and $\frac{u'}{v'}$ are consecutive Farey fractions such that $\frac{a}{q} \leq \frac{u}{v} < \frac{u'}{v'} \leq \frac{a'}{q'}$. So by replacing A with a power of A , we may assume that $A \in H$. We know there exist $a_0, b_0, c_0, d_0 \in \mathbb{N}$ such that $\begin{pmatrix} a_0 & b_0 \\ -c_0 & -d_0 \end{pmatrix} H \in M$. We then have

$$\begin{aligned} \begin{pmatrix} a_0 & b_0 \\ -c_0 & -d_0 \end{pmatrix} H &= \begin{pmatrix} a_0 & b_0 \\ -c_0 & -d_0 \end{pmatrix} \begin{pmatrix} q & a \\ q' & a' \end{pmatrix} H \\ &= \begin{pmatrix} a_0 q + b_0 q' & a_0 a + b_0 a' \\ -c_0 q - d_0 q' & -c_0 a - d_0 a' \end{pmatrix} H. \end{aligned}$$

Notice that $\frac{a}{q} \leq \frac{c_0 a + d_0 a'}{c_0 q + d_0 q'} < \frac{a_0 a + b_0 a'}{a_0 q + b_0 q'} \leq \frac{a'}{q'}$, and thus $\frac{c_0 a + d_0 a'}{c_0 q + d_0 q'} < \frac{a_0 a + b_0 a'}{a_0 q + b_0 q'}$ are consecutive Farey fractions in I' such that

$$\begin{pmatrix} a_0 q + b_0 q' & a_0 a + b_0 a' \\ -c_0 q - d_0 q' & -c_0 a - d_0 a' \end{pmatrix} H \in M.$$

We conclude that $\frac{c_0 a + d_0 a'}{c_0 q + d_0 q'} \in \mathcal{F}_{I', M}(Q)$ for $Q \geq c_0 q + d_0 q'$. \square

4. A POINCARÉ SECTION FOR G/H

In the same way the properties of Ω' as a Poincaré section of the horocycle flow on G/Γ were used in [3] to deduce many consequences for the gaps in $(\mathcal{F}(Q))$, we now find a new space Ω_M that is a Poincaré section of the horocycle flow on G/H which can be used to analyze the gaps in $\mathcal{F}_M(Q)$. One step toward this goal is to lift the Poincaré section Ω' to G/H via the natural projection $\pi : G/H \rightarrow G/\Gamma$. In order to do this, we use the work of Fisher and Schmidt [12] on the behavior of a lifted Poincaré section for the geodesic flow from $F \backslash \mathrm{PSL}(2, \mathbb{R})$, $F \subseteq \mathrm{PSL}(2, \mathbb{R})$ being a Fuchsian group of finite covolume, to a finite cover $F' \backslash \mathrm{PSL}(2, \mathbb{R})$ of $F \backslash \mathrm{PSL}(2, \mathbb{R})$. In particular, we apply [12, Lemma 2, Theorem 3] to the finite cover $\pi : G/H \rightarrow G/\Gamma$, lifting the Poincaré section Ω' to $\Omega'' := \pi^{-1}(\Omega')$. In applying Theorem 3, we make the slight modifications of working in the left coset space instead of the right coset space, replacing the geodesic flow with the horocycle flow, and allowing the possibility that $-I \notin H$, so that G/H is not necessarily of the form $\mathrm{PSL}(2, \mathbb{R})/F'$. We summarize the results we need from this application in the following theorem:

Theorem 2. *The set*

$$\Omega'' := \pi^{-1}(\Omega') = P\Gamma H/H = \{p_{a,b}\gamma H : p_{a,b} \in P, \gamma \in \Gamma\}$$

is a Poincaré section for the action of N on G/H with first return time function $r \circ \pi : \Omega'' \rightarrow \mathbb{R}$ and first return map $R' : \Omega'' \rightarrow \Omega''$ such that, for all $p_{a,b} \in \Omega$ and $\gamma \in \Gamma$,

$$\begin{aligned} R'(p_{a,b}\gamma H) &= h_{r(\pi(p_{a,b}\gamma H))} p_{a,b}\gamma H \\ &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{ab} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \lfloor \frac{1+a}{b} \rfloor \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \lfloor \frac{1+a}{b} \rfloor \end{pmatrix}^{-1} \gamma H \\ &= \begin{pmatrix} b & \lfloor \frac{1+a}{b} \rfloor b - a \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} \lfloor \frac{1+a}{b} \rfloor & 1 \\ -1 & 0 \end{pmatrix} \gamma H. \end{aligned} \quad (4.1)$$

Identify Ω'' with $P \times \Gamma/H$ via the correspondence $p_{a,b}\gamma H \leftrightarrow (p_{a,b}, \gamma H)$, and let $\mu_{\Omega''}$ be the measure on Ω'' corresponding in this way to the product measure on $P \times \Gamma/H$ of $2da db$ with the counting measure on Γ/H . Then identify G/H with $\{(x, s) \in \Omega'' \times \mathbb{R} : 0 \leq s < (r \circ \pi)(x)\}$ via $(x, s) \leftrightarrow h_s x$. Then the Haar measure $\mu_{G/H}$ on G/H , normalized so that $\mu_{G/H}(G/H) = \frac{\pi^2}{3}[\Gamma : H]$, is given by $d\mu_{G/H} = d\mu_{\Omega''} ds$.

Our next step is to find a correspondence of $\mathcal{F}(Q)$ with points in Ω'' , analogous to that of $\mathcal{F}(Q)$ with \mathcal{T} given in Section 2. So suppose that $\frac{a}{q} < \frac{a'}{q'} < \frac{a''}{q''}$ are consecutive fractions in $\mathcal{F}(Q)$. Then letting $K = \lfloor \frac{Q+q}{q'} \rfloor$ and noting that $a'' = Ka' - a$ and $q'' = Kq' - q$, by (4.1) we have

$$\begin{aligned} R' \left(\begin{pmatrix} \frac{q}{Q} & \frac{q'}{Q} \\ 0 & \frac{q}{q'} \end{pmatrix} \begin{pmatrix} q' & a' \\ -q & -a \end{pmatrix} H \right) &= \begin{pmatrix} \frac{q'}{Q} & \frac{Kq' - q}{Q} \\ 0 & \frac{q}{q'} \end{pmatrix} \begin{pmatrix} Kq' - q & Ka' - a \\ -q' & -a' \end{pmatrix} H \\ &= \begin{pmatrix} \frac{q'}{Q} & \frac{q''}{Q} \\ 0 & \frac{q}{q'} \end{pmatrix} \begin{pmatrix} q'' & a'' \\ -q' & -a' \end{pmatrix} H. \end{aligned}$$

Therefore, if we associate each fraction $\frac{a}{q} \in \mathcal{F}(Q)$ to the element

$$W_{H,Q}\left(\frac{a}{q}\right) = \begin{pmatrix} \frac{q}{Q} & \frac{q'}{Q} \\ 0 & \frac{Q}{q} \end{pmatrix} \begin{pmatrix} q' & a' \\ -q & -a \end{pmatrix} H$$

of Ω'' , where $\frac{a'}{q'}$ is the element succeeding $\frac{a}{q}$ in $\mathcal{F}(Q)$, then $R'(W_{H,Q}(\frac{a}{q})) = W_{H,Q}(\frac{a'}{q'})$. The map $W_{H,Q}$ gives the correspondence of $\mathcal{F}(Q)$ with Ω'' that we are seeking, and $W_{H,Q}$ sends a fraction in $\mathcal{F}(Q)$ to

$$\Omega_M := PM = \{pm_i H : p \in P, m_i H \in M\}$$

if and only if the fraction is in $\mathcal{F}_M(Q)$. The set Ω_M is the Poincaré section in G/H we set out to find at the beginning of this section.

To see that Ω_M is in fact a Poincaré section for the horocycle flow on G/H , notice that the set

$$h_{[-1,0]}\Omega_M = \{h_s pm_i H : s \in [-1,0], pm_i H \in \Omega_M\} \subseteq G/H$$

has positive $\mu_{G/H}$ -measure. Now by the Howe-Moore theorem [14], the horocycle flow is ergodic, and in fact mixing, on G/H . Thus $\mu_{G/H}$ -a.e. $x \in G/H$ is sent to $h_{[-1,0]}\Omega_M$ by $\{h_s : s > 0\}$. Clearly all of $h_{[-1,0]}\Omega_M$ is sent to Ω_M by $\{h_s : s \geq 0\}$, and so a.e. $x \in G/H$ is sent to Ω_M by $\{h_s : s > 0\}$. The discreteness of $\{s \in \mathbb{R} : h_s x \in \Omega_M\}$ for a.e. $x \in G/H$ follows from the fact that $\Omega_M \subseteq \Omega''$. This proves that Ω_M is a Poincaré section for the action of N on G/H .

Let $r_M : \Omega_M \rightarrow (0, \infty]$ be the first return time function $r_M(x) = \min\{s > 0 : h_s x \in \Omega_M\}$ and R_M be the first return map on Ω_M defined by $R_M(x) = h_{r_M(x)}(x)$. Here we note that $d\mu_{G/H} = d(\mu_{\Omega''}|_{\Omega_M}) ds$, where we identify G/H with $\{(x, s) \in \Omega_M \times \mathbb{R} : 0 \leq s < r_M(x)\}$ by $h_s x \leftrightarrow (x, s)$.

Now that we have identified a Poincaré section in G/H that can parameterize $\mathcal{F}_M(Q)$ in the same way that $\mathcal{T} \cong \Omega'$ parameterizes $\mathcal{F}(Q)$, we wish to see if information about the gaps in $\mathcal{F}_M(Q)$ can be deduced from Ω_M . So let $\gamma_i \in \mathcal{F}_M(Q)$ such that there are $i' > i$ with $\gamma_{i'} \in \mathcal{F}_M(Q)$. Since $\Omega_M \subseteq \Omega''$, for each $x \in \Omega_M$ in which $R_M(x)$ is defined, $R_M(x) = R'^j(x)$ for some $j \in \mathbb{N}$. So

$$R_M(W_{H,Q}(\gamma_i)) = R'^j(W_{H,Q}(\gamma_i)) = W_{H,Q}(\gamma_{i+j}),$$

where $j \in \mathbb{N}$ is the least element such that $W_{H,Q}(\gamma_{i+j}) \in \Omega_M$, i.e., $\gamma_{i+j} \in \mathcal{F}_M(Q)$. We then have

$$\begin{aligned} r_M(W_{H,Q}(\gamma_i)) &= \sum_{i'=0}^{j-1} (r \circ \pi)(R'^{i'}(W_{H,Q}(\gamma_i))) = \sum_{i'=0}^{j-1} \frac{Q^2}{q_{i+i'} q_{i+i'+1}} \\ &= Q^2(\gamma_{i+j} - \gamma_i). \end{aligned} \tag{4.2}$$

So just as the return time function r on Ω' contained information about the gaps in $\mathcal{F}(Q)$, r_M contains information about the gaps in $\mathcal{F}_M(Q)$.

Let $N_{I,M}(Q) = \#\mathcal{F}_{I,M}(Q) - 1$ and

$$\mathcal{F}_{I,M}(Q) = \{\alpha_0 < \alpha_1 < \dots < \alpha_{N_{I,M}(Q)}\}.$$

For notational convenience, we suppress the dependence of the α_i on Q . Then define the measure $\rho_{Q,I,M}$ on Ω_M by

$$\rho_{Q,I,M} = \frac{1}{N_{I,M}(Q)} \sum_{i=0}^{N_{I,M}(Q)-1} \delta_{W_{H,Q}(\alpha_i)}.$$

By (4.2), we have

$$\frac{\#\{0 \leq i \leq N_{I,M}(Q) - 1 : Q^2(\alpha_{i+1} - \alpha_i) \in [0, c]\}}{N_{I,M}(Q)} = \rho_{Q,I,M}(r_M^{-1}[0, c]) \quad (4.3)$$

for all $c \geq 0$. To show that the limit of the left side, and hence the right side, exists, we prove in Section 5 that the boundary $\partial(r_M^{-1}[0, c])$ of the set $r_M^{-1}[0, c]$ has $\mu_{\Omega''}$ -measure 0, and then prove in Section 6 that $(\rho_{Q,I,M})$ converges weakly to $\frac{1}{\#M} \mu_{\Omega''}|_{\Omega_M}$ as $Q \rightarrow \infty$. These results imply that

$$\lim_{Q \rightarrow \infty} \rho_{Q,I,M}(r_M^{-1}[0, c]) = \frac{1}{\#M} \mu_{\Omega''}(r_M^{-1}[0, c])$$

by the Portmanteau theorem (see [6]). Note that the right side of (4.3) is not the relevant limit to prove the existence of the limiting gap measure for $(\mathcal{F}_{I,M}(Q))$. However, we have $\alpha_{N_{I,M}(Q)} - \alpha_0 \rightarrow |I|$ as $Q \rightarrow \infty$ by the density of $\bigcup_{Q \in \mathbb{N}} \mathcal{F}_M(Q)$ in $[0, 1]$, and we show that $N_{I,M}(Q) \sim \frac{|I|(\#M)Q^2}{\mu_{G/H}(G/H)} = \frac{3|I|(\#M)Q^2}{\pi^2[\Gamma:H]}$ in the course of our work in Section 6. It then follows that the limiting gap measure $\nu_{I,M}$ exists and satisfies

$$\nu_{I,M}([0, c]) = \frac{1}{\#M} \mu_{\Omega''} \left(r_M^{-1} \left[0, \frac{\pi^2[\Gamma:H]}{3(\#M)} c \right] \right) \quad (4.4)$$

for all $c \geq 0$. Another corollary is that $\lim_{Q \rightarrow \infty} \frac{N_{I,M}(Q)}{N_{[0,1],M}(Q)} = |I|$ for every subinterval $I \subseteq [0, 1]$, implying that $(\mathcal{F}_M(Q))$ becomes equidistributed in $[0, 1]$ as $Q \rightarrow \infty$.

5. THE RETURN TIME FUNCTION r_M

In this section, we prove important properties of the first return time function r_M of the Poincaré section Ω_M . In particular, we prove that $\mu_{\Omega''}(\partial(r_M^{-1}[0, c])) = 0$ for every $c \geq 0$, and the function $F_M : [0, \infty) \rightarrow [0, 1]$ defined by

$$F_M(c) = \frac{1}{\#M} \mu_{\Omega''}(r_M^{-1}[0, c])$$

has a continuous, piecewise real-analytic derivative. These results, together with our work in Section 6, proves that the limiting gap measure $\nu_{I,M}$ exists and has a continuous and piecewise real-analytic density. To do this, we show that r_M is a piecewise rational function, viewing each component Pm_iH/H of Ω_M as a copy of the Farey triangle \mathcal{T} by the correspondence $p_{a,b}m_iH \leftrightarrow (a, b)$, and that the region in a given component Pm_iH/H over which r_M is equal to a certain rational function is a polygon. This allows us to say that $r_M^{-1}[0, c]$ is the union of regions, each being obtained by intersecting a polygon with a region bounded below by a hyperbola which depends in a smooth way on c . Specifically, we show that $r_M^{-1}[0, c]$ is a finite union of these regions, which grants us the properties of F_M we seek.

5.1. r_M is piecewise rational. First let $p_{a,b} \in P$ and $m_iH \in M$ so that $p_{a,b}m_iH \in \Omega_M$, and suppose $s > 0$. We have $h_s p_{a,b}m_iH \in \Omega_M$ if and only if there exist $p_{c,d} \in P$ and $m_jH \in M$ such

that $h_s p_{a,b} m_i H = p_{c,d} m_j H$. This means there exists $h \in H$ such that $h_s p_{a,b} m_i h m_j^{-1} = p_{c,d}$. Letting $m_i h m_j^{-1} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, this equality is

$$\begin{pmatrix} ac_1 + bc_3 & ac_2 + bc_4 \\ a^{-1}c_3 - s(ac_1 + bc_3) & a^{-1}c_4 - s(ac_2 + bc_4) \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}.$$

Thus $h_s p_{a,b} m_i H \in \Omega_M$ if and only if there exists $\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in \bigcup_{j=1}^k m_i H m_j^{-1}$ such that $(ac_1 + bc_3, ac_2 + bc_4) \in \mathcal{T}$ and $s = \frac{c_3}{a(ac_1 + bc_3)} = \frac{1}{a(b + \frac{c_1}{c_3}a)}$. Note that the latter conditions and $s > 0$ imply that $c_3 > 0$, and hence $c_1 \leq 0$ since $ac_1 + bc_3 \leq 1$ and $a + b > 1$. In particular, if $r_M(p_{a,b} m_i H) < \infty$, then $r_M(p_{a,b} m_i H) = \frac{1}{a(b + \frac{c_1}{c_3}a)}$, where $\frac{c_1}{c_3}$ is the greatest fraction such that $c_1 \leq 0$, $c_3 > 0$, and there exists $\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in \bigcup_{j=1}^k m_i H m_j^{-1}$ with $(ac_1 + bc_3, ac_2 + bc_4) \in \mathcal{T}$. We have thus proven the following result:

Proposition 1. *The function r_M is a piecewise rational function. Specifically,*

$$r_M = \min_{C \in \bigcup_{i,j=1}^k m_i H m_j^{-1}} f_C,$$

where for each $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in \bigcup_{i,j=1}^k m_i H m_j^{-1}$, $f_C : \Omega_M \rightarrow [0, \infty]$ is defined by

$$f_C(p_{a,b} m_i H) = \begin{cases} \frac{1}{a(b + \frac{c_1}{c_3}a)} & \text{if } C \in \bigcup_{j=1}^k m_i H m_j^{-1}, c_3 > 0, (ac_1 + bc_3, ac_2 + bc_4) \in \mathcal{T} \\ \infty & \text{otherwise.} \end{cases}$$

Next, for a given fraction $\frac{c_1}{c_3}$ with $c_1 \leq 0$ and $c_3 > 0$, we wish to better understand the region

$$R_{-c_1/c_3} := \left\{ p_{a,b} m_i H \in \Omega_M : r_M(p_{a,b} m_i H) = \frac{1}{a(b + \frac{c_1}{c_3}a)} \right\}.$$

In particular, we want to prove the following technical result, which is a great aid in showing that $\mu_{\Omega''}(\partial(r_M^{-1}[0, c])) = 0$ for $c \geq 0$ and that F'_M is continuous and piecewise analytic.

Proposition 2. *For each $i \in \{1, \dots, k\}$, $R_{-c_1/c_3} \cap P m_i H / H$ is either empty or a polygon, viewing $P m_i H / H$ as the Farey triangle \mathcal{T} .*

Proof. We first examine the region

$$R_C := \left\{ p_{a,b} m_i H \in \Omega_M : (ac_1 + bc_3, ac_2 + bc_4) \in \mathcal{T}, C \in \bigcup_{j=1}^k m_i H m_j^{-1} \right\}$$

for a given $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in \bigcup_{i,j=1}^k m_i H m_j^{-1}$. Note that R_{-c_1/c_3} is a subset of the union of all $R_{C'}$ such that $C' \in \bigcup_{i,j=1}^k m_i H m_j^{-1}$ is a matrix having $\begin{pmatrix} c_1 \\ c_3 \end{pmatrix}$ as its first column. If $c_4 = 0$, then $C = \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix}$ and for a given index i , $R_C \cap P m_i H / H$ is a subset of $\{p_{a,b} m_i H : (ac_1 + c_3, -a) \in \mathcal{T}\} = \emptyset$. Thus R_C is empty in this case. Next, suppose $c_4 \leq -1$. Then $R_C \cap P m_i H / H$ consists of the elements $p_{a,b} m_i H$ which must satisfy

$$-\frac{c_1}{c_3}a < b \leq \frac{1-c_1a}{c_3} \quad \text{and} \quad \frac{1-c_2a}{c_4} \leq b < -\frac{c_2}{c_4}a.$$

However, we have $c_1 c_4 - c_2 c_3 = 1$, which implies that $\frac{c_1}{c_3} - \frac{c_2}{c_4} = \frac{1}{c_3 c_4} < 0$, and hence $-\frac{c_2}{c_4} < -\frac{c_1}{c_3}$. Thus the above conditions on $p_{a,b} m_i H$ cannot be satisfied, and therefore $R_C = \emptyset$. So R_C is nonempty only when $c_1 \leq 0$ and $c_3, c_4 \geq 1$, which then implies that $c_2 = \frac{c_1 c_4 - 1}{c_3} < 0$.

As a result of our work above, we reset notation so that $C = \begin{pmatrix} -c_1 & -c_2 \\ c_3 & c_4 \end{pmatrix}$ and we assume that $c_1 \geq 0$ and $c_2, c_3, c_4 \geq 1$. Now a point $(a, b) \in \mathcal{T}$ satisfies $(-ac_1 + bc_3, -ac_2 + bc_4) \in \mathcal{T}$ if and only if

$$\frac{c_1}{c_3}a < b \leq \frac{1+c_1a}{c_3}, \quad \frac{c_2}{c_4}a < b \leq \frac{1+c_2a}{c_4}, \quad \text{and} \quad b > \frac{1+(c_1+c_2)a}{c_3+c_4}.$$

Since $\frac{c_1}{c_3}a < \frac{c_2}{c_4}a \leq \frac{1+(c_1+c_2)a}{c_3+c_4}$ for $a \in (0, 1]$, the above conditions reduce to $\frac{1+(c_1+c_2)a}{c_3+c_4} < b \leq \min\{\frac{1+c_1a}{c_3}, \frac{1+c_2a}{c_4}\}$. We therefore have

$$R_C = \left\{ p_{a,b}m_iH \in \Omega_M : \begin{array}{l} \frac{1+(c_1+c_2)a}{c_3+c_4} < b \leq \min\{\frac{1+c_1a}{c_3}, \frac{1+c_2a}{c_4}\} \\ C \in \bigcup_{j=1}^k m_iHm_j^{-1} \end{array} \right\}.$$

Since M is closed under left multiplication by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\bigcup_{j=1}^k m_iHm_j^{-1}$ is closed under right multiplication by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for each $i \in \{1, \dots, k\}$. So if $C \in \bigcup_{j=1}^k m_iHm_j^{-1}$, then for every $n \in \mathbb{N}$,

$$C \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -c_1 & -nc_1 - c_2 \\ c_3 & nc_3 + c_4 \end{pmatrix} \in \bigcup_{j=1}^k m_iHm_j^{-1}.$$

We have

$$R_{C\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}} = \left\{ p_{a,b}m_iH \in \Omega_M : \begin{array}{l} \frac{1+((n+1)c_1+c_2)a}{(n+1)c_3+c_4} < b \leq \frac{1+(nc_1+c_2)a}{nc_3+c_4} \\ C \in \bigcup_{j=1}^k m_iHm_j^{-1} \end{array} \right\},$$

noting that $\frac{1+(nc_1+c_2)a}{nc_3+c_4} < \frac{1+c_3a}{c_4}$. Thus for each i such that $C \in \bigcup_{j=1}^k m_iHm_j^{-1}$, the regions $\{R_{C\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}} \cap Pm_iH/H : n \in \mathbb{N}\}$ paste together to form

$$\left\{ p_{a,b}m_iH \in Pm_iH/H : \frac{c_1}{c_3}a < b \leq \min\left\{\frac{1+c_1a}{c_3}, \frac{1+c_2a}{c_4}\right\} \right\}$$

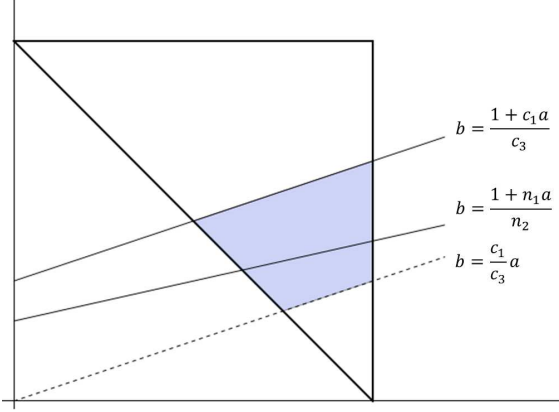
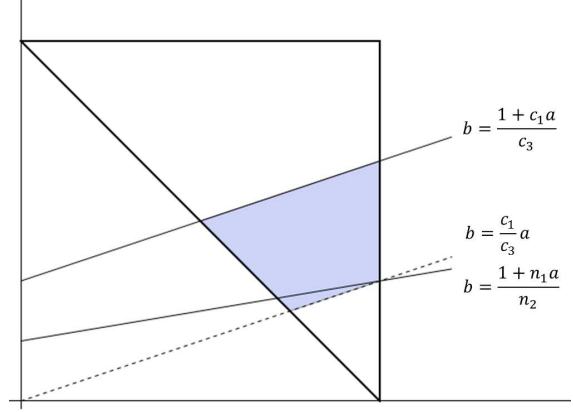
since the sequence $(\frac{1+(nc_1+c_2)a}{nc_3+c_4})$ decreases to $\frac{c_1}{c_3}a$ for each $a \in (0, 1]$. Therefore, if we assume that $\frac{c_{2,i}}{c_{4,i}}$ is the largest fraction such that $\begin{pmatrix} -c_1 & -c_{2,i} \\ c_3 & c_{4,i} \end{pmatrix} \in \bigcup_{j=1}^k m_iHm_j^{-1}$, then

$$R_{c_1/c_3} \cap Pm_iH/H \subseteq \left\{ p_{a,b}m_iH : \frac{c_1}{c_3}a < b \leq \min\left\{\frac{1+c_1a}{c_3}, \frac{1+c_{2,i}a}{c_{4,i}}\right\} \right\}.$$

We use $R_{c_1/c_3}^{(i)}$ to denote the set on the right for each $c_1 \geq 0$, $c_3 \geq 1$, and $i \in \{1, \dots, k\}$ such that $\begin{pmatrix} -c_1 \\ c_3 \end{pmatrix}$ is the first column of a matrix in $\bigcup_{j=1}^k m_iHm_j^{-1}$, and $\frac{c_{2,i}}{c_{4,i}}$ ($c_{2,i}, c_{4,i} \geq 1$) is the largest fraction with $\begin{pmatrix} -c_1 & -c_{2,i} \\ c_3 & c_{4,i} \end{pmatrix} \in \bigcup_{j=1}^k m_iHm_j^{-1}$. If c_1 , c_3 , and i do not satisfy these conditions, we let $R_{c_1/c_3}^{(i)} = \emptyset$. Then for all c_1 and c_3 , we have

$$R_{c_1/c_3} = \bigcup_{i=1}^k \left(R_{c_1/c_3}^{(i)} \setminus \bigcup_{s \in \mathbb{Q}, 0 \leq s < \frac{c_1}{c_3}} R_s^{(i)} \right).$$

Assume that c_1 , c_3 , and i are such that $R_{c_1/c_3}^{(i)} \neq \emptyset$. In order to show that $R_{c_1/c_3} \cap Pm_iH/H$ is either empty or a polygon, it is sufficient to prove that $R_{c_1/c_3} \cap Pm_iH/H$ can be written in the form $R_{c_1/c_3}^{(i)} \setminus (\bigcup_{\ell=1}^n R_{s_\ell}^{(i)})$ for some $s_\ell \in \mathbb{Q}$. Note that if $\frac{c_1}{c_3} = 0$, then $R_{c_1/c_3} \cap Pm_iH/H = R_0^{(i)}$, which is a triangle if nonempty. So assume that $\frac{c_1}{c_3} > 0$. We now consider two cases.

FIGURE 1. R_{c_1/c_3} in Case 1FIGURE 2. R_{c_1/c_3} in Case 2

Case 1. There exists $s \in \mathbb{Q}$ with $s < \frac{c_1}{c_3}$ such that $R_s^{(i)}$ contains the lower-right border $\{p_{a,b}m_iH : b = \frac{c_1}{c_3}a\}$ of $R_{c_1/c_3}^{(i)}$ in its interior. Then clearly there exists $N \in \mathbb{N}$ such that

$$R_{c_1/c_3}^{(i)} \setminus R_s^{(i)} \subseteq \left\{ p_{a,b}m_iH : \frac{1}{N} + \frac{c_1}{c_3}a < b \leq \frac{1+c_1a}{c_3} \right\}.$$

If there is another $\frac{c'_1}{c_3} \in \mathbb{Q}$ with $\frac{c'_1}{c_3} < \frac{c_1}{c_3}$ such that $R_{c'_1/c_3}^{(i)}$ intersects $R_{c_1/c_3}^{(i)} \setminus R_s^{(i)}$, then there exists $(a, b) \in \mathcal{T}$ such that $\frac{1}{N} + \frac{c_1}{c_3}a < b \leq \frac{1+c'_1a}{c_3}$. This implies that $(\frac{c_1}{c_3} - \frac{c'_1}{c_3})a < \frac{1}{c_3} - \frac{1}{N}$, which can only hold if $c'_3 < N$. Thus there are finitely many $s' \in \mathbb{Q}$ such that $R_{s'}^{(i)}$ intersects $R_{c_1/c_3}^{(i)} \setminus R_s^{(i)}$. So

$$R_{c_1/c_3} \cap Pm_iH/H = R_{c_1/c_3}^{(i)} \setminus \left(R_s^{(i)} \cup \bigcup_{\ell=1}^n R_{s_\ell}^{(i)} \right)$$

for some $s_\ell \in \mathbb{Q}$, completing the proof that $R_{c_1/c_3} \cap Pm_iH/H$ is a polygon in this case.

Case 2. There exists no $s \in \mathbb{Q}$ with $s < \frac{c_1}{c_3}$ such that $R_s^{(i)}$ contains the lower-right border of $R_{c_1/c_3}^{(i)}$ in its interior. Let $n_1 \geq 0$ and $n_2 \geq 1$ be integers such that $\{p_{a,b}m_iH : b = \frac{1+n_1a}{n_2}\}$ is an upper border for $R_s^{(i)}$ for some $s \in \mathbb{Q}$ with $s < \frac{c_1}{c_3}$. If $\frac{n_1}{n_2} = \frac{c_1}{c_3}$, then the line $\{(a, b) \in \mathbb{R}^2 : b = \frac{1+n_1a}{n_2}\}$ is above and parallel to $\{(a, b) \in \mathbb{R}^2 : b = \frac{c_1}{c_3}a\}$, in which case the lower-right border of $R_{c_1/c_3}^{(i)}$ is contained in the interior of $R_s^{(i)}$, a contradiction. So $\frac{n_1}{n_2} \neq \frac{c_1}{c_3}$, and $\{(a, b) \in \mathbb{R}^2 : b = \frac{1+n_1a}{n_2}\}$ intersects $\{(a, b) \in \mathbb{R}^2 : b = \frac{c_1}{c_3}a\}$ at the point $(\frac{c_3}{n_2c_1 - n_1c_3}, \frac{c_1}{n_2c_1 - n_1c_3})$. If $\frac{n_1}{n_2} > \frac{c_1}{c_3}$, then since $\frac{c_3}{n_2c_1 - n_1c_3} < 0$ and the slope of $b = \frac{1+n_1a}{n_2}$ is greater than that of $b = \frac{c_1}{c_3}a$, the lower-right border of $R_{c_1/c_3}^{(i)}$ is again contained in the interior of $R_s^{(i)}$, another contradiction.

So we have $\frac{n_1}{n_2} < \frac{c_1}{c_3}$, and furthermore, the intersection point $(\frac{c_3}{n_2c_1 - n_1c_3}, \frac{c_1}{n_2c_1 - n_1c_3})$ cannot be above or to the right of \mathcal{T} . If the intersection point is in the interior of \mathcal{T} or is on or below its border $b = 1 - a$, then $R_{c_1/c_3}^{(i)} \setminus R_s^{(i)}$ contains a set of the form

$$\{p_{a,b}m_iH : b - \frac{c_1}{c_3}a \in (0, \epsilon), a \in (t_1, t_2)\}$$

where $\epsilon, t_1, t_2 \in (0, 1)$ and $t_1 < t_2$. Since $\frac{1}{a(b-\frac{c_1}{c_3}a)}$ is not $\mu_{\Omega''}$ -integrable over the above set for any $\epsilon, t_1, t_2 \in (0, 1)$ with $t_1 < t_2$ and r_M is $\mu_{\Omega''}$ -integrable, the lower-right border of $R_{c_1/c_3}^{(i)}$ is contained in $\bigcup_{s \in \mathbb{Q}, 0 \leq s < \frac{c_1}{c_3}} R_s^{(i)}$. Since the set $\{(\frac{c_3}{n}, \frac{c_1}{n}) : n \in \mathbb{N}\} \cap \mathcal{T}$ of possible intersection points in \mathcal{T} of $b = \frac{c_1}{c_3}a$ with a line $b = \frac{1+n_1a}{n_2}$ corresponding to the upper-left border of a set $R_s^{(i)}$ is finite, there must be some $s \in \mathbb{Q}$ with $s < \frac{c_1}{c_3}$ such that $R_s^{(i)}$ contains the lower-right border of $R_{c_1/c_3}^{(i)}$, and in this case, the line determining the upper-left border of $R_s^{(i)}$, say $b = \frac{1+n_1a}{n_2}$, intersects $b = \frac{c_1}{c_3}a$ at the border of \mathcal{T} at the line $a = 1$ or $b = 1$.

Now let $s' \in \mathbb{Q}$ with $s' < \frac{c_1}{c_3}$ such that the upper-left border of $R_{s'}^{(i)}$ is determined by $b = \frac{1+n'_1a}{n'_2}$. Since $R_{s'}^{(i)}$ does not contain the lower-right border of $R_{c_1/c_3}^{(i)}$, $\frac{n'_1}{n'_2} < \frac{c_1}{c_3}$ and $b = \frac{1+n'_1a}{n'_2}$ intersects $b = \frac{c_1}{c_3}a$ at or to the left of the intersection point of $b = \frac{1+n_1a}{n_2}$ with $b = \frac{c_1}{c_3}a$. If $\frac{n'_1}{n'_2} > \frac{n_1}{n_2}$, then it is clear that the line $b = \frac{1+n'_1a}{n'_2}$ passes under the set $\{(a, b) \in \mathcal{T} : b > \max\{\frac{1+n_1a}{n_2}, \frac{c_1}{c_3}a\}\}$, and thus $R_{s'}^{(i)}$ does not intersect $R_{c_1/c_3}^{(i)} \setminus R_s^{(i)}$. If $\frac{n'_1}{n'_2} < \frac{n_1}{n_2}$ and $R_{s'}^{(i)}$ does intersect $R_{c_1/c_3}^{(i)} \setminus R_s^{(i)}$, then there exists $(a, b) \in \mathcal{T}$ such that $\frac{1+n_1a}{n_2} < b \leq \frac{1+n'_1a}{n'_2}$, implying that $(\frac{n_1}{n_2} - \frac{n'_1}{n'_2})a < \frac{1}{n'_2} - \frac{1}{n_2}$. This inequality holds only if $n'_2 < n_2$. This shows that there are finitely many $s' \in \mathbb{Q}$ such that $R_{s'}^{(i)}$ intersects $R_{c_1/c_3}^{(i)} \setminus R_s^{(i)}$, and thus completes the proof that $R_{c_1/c_3} \cap Pm_i H/H$ is a polygon. \square

So by our work above, we have proven that r_M is a piecewise rational function on Ω_M , and for a given $\frac{c_1}{c_3} \in \mathbb{Q}$, the region R_{c_1/c_3} over which $r_M(p_{a,b}m_i H) = \frac{1}{a(b-\frac{c_1}{c_3}a)}$ is either empty or a union of polygons, one polygon being in each component $Pm_i H/H$ such that $R_{c_1/c_3} \cap Pm_i H/H \neq \emptyset$.

5.2. The boundary of $r_M^{-1}[0, c]$ has measure 0. Next, we prove that for a given $c > 0$, the boundary of $r_M^{-1}[0, c]$ has $\mu_{\Omega''}$ -measure 0. First notice that r_M is continuous $\mu_{\Omega''}$ -a.e. Since the set $\overline{r_M^{-1}[0, c]} \setminus r_M^{-1}[0, c]$ contains only points of discontinuity of r_M , it has $\mu_{\Omega''}$ -measure 0. Next, for a given $s \in \mathbb{Q}$, define $f_s : \Omega_M \rightarrow [0, \infty]$ so that

$$f_s(p_{a,b}m_i H) = \begin{cases} \frac{1}{a(b-sa)} & \text{for all } p_{a,b}m_i H \in R_s \\ \infty & \text{otherwise.} \end{cases}$$

We then have

$$\begin{aligned} r_M^{-1}[0, c] \setminus (r_M^{-1}[0, c])^o &= \bigcup_{s \in \mathbb{Q}} f_s^{-1}[0, c] \setminus \left(\bigcup_{s \in \mathbb{Q}} f_s^{-1}[0, c] \right)^o \\ &\subseteq \bigcup_{s \in \mathbb{Q}} f_s^{-1}[0, c] \setminus \left(\bigcup_{s \in \mathbb{Q}} (f_s^{-1}[0, c])^o \right) \\ &\subseteq \bigcup_{s \in \mathbb{Q}} ((f_s^{-1}[0, c]) \setminus (f_s^{-1}[0, c])^o). \end{aligned}$$

Each set $f_s^{-1}[0, c]$ is either empty or a finite union of sets of the form

$$\left\{ p_{a,b} m_i H \in P m_i H : \frac{c_1}{c_3} a < b \leq \min\left\{ \frac{1 + c_1 a}{c_3}, \frac{1 + c_{2,i} a}{c_{4,i}} \right\}, b \geq \frac{c_1}{c_3} a + \frac{1}{ca} \right\}.$$

So $(f_s^{-1}[0, c]) \setminus (f_s^{-1}[0, c])^o$ is essentially a set of finitely many line and curve segments for each $s \in \mathbb{Q}$. Hence it is clear that $r_M^{-1}[0, c] \setminus (r_M^{-1}[0, c])^o$ is of $\mu_{\Omega''}$ -measure 0. As a consequence,

$$\partial(r_M^{-1}[0, c]) = \overline{r_M^{-1}[0, c] \setminus (r_M^{-1}[0, c])^o} = \overline{(r_M^{-1}[0, c] \setminus r_M^{-1}[0, c])} \cup (r_M^{-1}[0, c] \setminus (r_M^{-1}[0, c])^o)$$

has $\mu_{\Omega''}$ -measure 0. This, along with Section 6, proves the existence of the limiting gap measure $\nu_{I,M}$.

5.3. F'_M is continuous and piecewise real-analytic. Next, we want to prove that the function F_M has a continuous, piecewise real-analytic derivative. Note first that since $r_M^{-1}[0, c]$ is the disjoint union of the sets $f_s^{-1}[0, c]$, we have

$$F_M(c) = \frac{1}{\#M} \sum_{s \in \mathbb{Q}} \mu_{\Omega''}(f_s^{-1}[0, c]).$$

So it suffices to show that $c \mapsto \mu_{\Omega''}(f_s^{-1}[0, c])$ has a continuous, piecewise real-analytic derivative for every $s \in \mathbb{Q}$, and that for a given $c > 0$, there are at most finitely many $s \in \mathbb{Q}$ for which $f_s^{-1}[0, c]$ is nonempty.

The first claim is easy to see. Indeed, by triangulating the polygons that make up the region R_s , we can write $c \mapsto \mu_{\Omega''}(f_s^{-1}[0, c])$ as a finite sum of functions $g_T : [0, \infty) \rightarrow \mathbb{R}$ defined by

$$g_T(c) = 2\lambda(\{(a, b) \in T : b \geq sa + \frac{1}{ca}\}),$$

where $T \subseteq \mathcal{T}$ is a triangle and λ is the Lebesgue measure. It is then straightforward to show that each function g_T has a continuous, piecewise real-analytic derivative, implying that $c \mapsto \mu_{\Omega''}(f_s^{-1}[0, c])$ has the same property.

To prove the latter assertion, let $c > 0$ be given and suppose that there exists $\frac{c_1}{c_3} \in \mathbb{Q}$ such that $f_{c_1/c_3}^{-1}[0, c]$ is nonempty (assume $c_1 \geq 0$ and $c_3 \geq 1$). Then there is some index i such that $f_{c_1/c_3}^{-1}[0, c] \cap P m_i H / H \neq \emptyset$. We have

$$f_{c_1/c_3}^{-1}[0, c] \cap P m_i H / H \subseteq \left\{ p_{a,b} m_i H : \frac{c_1}{c_3} a < b \leq \frac{1 + c_1 a}{c_3}, b \geq \frac{c_1}{c_3} a + \frac{1}{ca} \right\},$$

and hence the latter set is nonempty. So there exists $p_{a,b} m_i H \in P m_i H / H$ such that $\frac{c_1}{c_3} a + \frac{1}{ca} \leq b \leq \frac{1 + c_1 a}{c_3}$, which implies that $c \geq \frac{c_3}{a}$. We have

$$\sup \left\{ a \in \mathbb{R} : (a, b) \in \mathcal{T}, \frac{c_1}{c_3} a < b \leq \frac{1 + c_1 a}{c_3} \right\} = \begin{cases} 1 & \text{if } \frac{c_1}{c_3} \leq 1 \\ \frac{c_3}{c_1} & \text{if } \frac{c_1}{c_3} > 1 \end{cases},$$

and therefore $c \geq c_3$ if $\frac{c_1}{c_3} \leq 1$ and $c \geq c_1$ if $\frac{c_1}{c_3} > 1$; i.e., $c \geq \max\{c_1, c_3\}$. There are clearly finitely many positive fractions $\frac{c_1}{c_3}$ satisfying this condition, and thus satisfying $f_{c_1/c_3}^{-1}([0, c]) \neq \emptyset$. This completes the proof that F'_M is continuous and piecewise real-analytic.

One more property of r_M we wish to mention is that for any $\epsilon > 0$, there exist bounded continuous functions $g_{1,\epsilon}, g_{2,\epsilon}, g_{3,\epsilon} : \Omega_M \rightarrow [0, \infty)$ such that $g_{1,\epsilon} \leq r_M$, $g_{2,\epsilon} \leq \frac{1}{r_M} \leq g_{3,\epsilon}$, and

$$\int_{\Omega_M} (r_M - g_{1,\epsilon}) d\mu_{\Omega''}, \int_{\Omega_M} \left(\frac{1}{r_M} - g_{2,\epsilon}\right) d\mu_{\Omega''}, \int_{\Omega_M} \left(g_{3,\epsilon} - \frac{1}{r_M}\right) d\mu_{\Omega''} < \epsilon.$$

This follows easily from the properties of r_M proven in Section 5.1, and the fact that $\frac{1}{r_M} \leq \frac{1}{r} \leq 1$.

5.4. The h -spacings and numerators of differences in $(\mathcal{F}_{I,M}(Q))$. For $h \in \mathbb{N}$ and $A = \{x_0 \leq x_1 \leq \dots \leq x_N\} \subseteq [0, 1]$, let $\mathbf{v}_{A,i,h} = (x_{i+j} - x_{i+j-1})_{j=1}^h \in \mathbb{R}^h$ for $i \in \{1, \dots, N-h\}$. We define the h -spacing distribution measure of A to be the measure $\nu_{A,h}$ on $[0, \infty)^h$ such that

$$\nu_{A,h} \left(\prod_{j=1}^h [0, c_j] \right) = \frac{1}{N} \# \left\{ x_i \in A : N \mathbf{v}_{A,i,h} \in \prod_{j=1}^h [0, c_j(x_N - x_0)] \right\}.$$

For an increasing sequence (A_n) of subsets of $[0, 1]$, we call the weak limit of $(\nu_{A_n,h})$, if it exists, the *limiting h -spacing measure* of (A_n) .

Upon $(r_M \circ R_M^j)(W_{H,Q}(\alpha_i)) = Q^2(\alpha_{i+j} - \alpha_{i+j-1})$, $i \in \{0, \dots, N_{I,M}(Q) - h\}$, we have

$$\frac{\#\{\alpha_i \in \mathcal{F}_{I,M}(Q) : Q^2 \mathbf{v}_{\mathcal{F}_{I,M}(Q),i,h} \in \prod_{j=1}^h [0, c_j]\}}{N_{I,M}(Q)} = \rho_{Q,I,M} \left(\bigcap_{j=1}^h (r_M \circ R_M^j)^{-1}[0, c_j] \right).$$

For a given $j \in \mathbb{N}$, the function $r_M \circ R_M^j$, like r_M itself, is piecewise rational and the domain in which $r_M \circ R_M^j$ is defined by a given rational function is a union of polygons. Indeed, it is easy to see from the facts in Section 5.1 that for any $i \in \mathbb{N}$, the i th return map of Ω_M satisfies these properties, and since $r_M \circ R_M^j$ is the difference between the j th and the $(j+1)$ st return maps of Ω_M , $r_M \circ R_M^j$ satisfies the same properties. As a consequence, the sets $(r_M \circ R_M^j)^{-1}[0, c_j]$ have boundaries of measure 0. Hence, by our work in Section 6, the limiting h -spacing measure $\nu_{I,M,h}$ of $(\mathcal{F}_{I,M}(Q))$ exists and satisfies

$$\nu_{I,M,h} \left(\prod_{j=1}^h [0, c_j] \right) = \frac{1}{\#M} \mu_{\Omega''} \left(\bigcap_{j=1}^h (r_M \circ R_M^j)^{-1} \left(\left[0, \frac{\pi^2[\Gamma : H]}{3(\#M)} c_j \right] \right) \right).$$

Lastly, note that if $\frac{a}{q} < \frac{b}{p}$ are consecutive elements in $\mathcal{F}_{I,M}(Q)$, $\frac{a'}{q'}$ succeeds $\frac{a}{q}$ in $\mathcal{F}(Q)$, and $W_{H,Q}(\frac{b}{p}) \in R_{c_1/c_3}$, then $p = -c_1q + c_3q'$ and

$$bq - ap = qp \left(\frac{b}{p} - \frac{a}{q} \right) = \frac{q(-c_1q + c_3q')}{Q^2} r_M \left(W_{H,Q} \left(\frac{a}{q} \right) \right) = \frac{q(-c_1q + c_3q')}{Q^2} \frac{Q^2}{q(q' - \frac{c_1}{c_3}q)} = c_3.$$

Using this fact and Section 6, one can show that for every $c_3 \in \mathbb{N}$,

$$\lim_{Q \rightarrow \infty} \frac{\#\{\frac{a}{q} < \frac{b}{q} \text{ consecutive in } \mathcal{F}_{I,M}(Q) : bq - ap = c_3\}}{N_{I,M}(Q)} = \frac{1}{\#M} \mu_{\Omega''} \left(\bigcup_{\substack{c_1/c_3 \in \mathbb{Q} \\ (c_1, c_3) = 1}} R_{c_1/c_3} \right),$$

recovering a result Badziahin and Haynes proved for the sequence $(\mathcal{F}_{Q,d})$ in [5].

6. THE CONVERGENCE $\rho_{Q,I,M} \rightarrow \mu_{\Omega''}|_{\Omega_M}$

In this section, we prove the weak convergence $\rho_{Q,I,M} \rightarrow \mu_{\Omega''}|_{\Omega_M}$, and hence complete the proof of Theorem 1. We first consider the measures $(\rho_{Q,I,M}^R)$ on G/H defined by $d\rho_{Q,I,M}^R = \frac{N_{I,M}(Q)}{Q^2} d\rho_{Q,I,M} ds$. In other words, $\rho_{Q,I,M}^R$ is a measure concentrated on segments of the horocycle flow connecting $W_{H,Q}(\alpha_i)$ to $W_{H,Q}(\alpha_{i+1})$ for $0 \leq i \leq N_{I,M}(Q) - 1$. These segments connect to give one segment from $W_{H,Q}(\alpha_0)$ to $W_{H,Q}(\alpha_{N_{I,M}(Q)})$. So for a bounded, measurable function $f : G/H \rightarrow \mathbb{R}$,

$$\int f d\rho_{Q,I,M}^R = \frac{1}{Q^2} \int_{Q^2\alpha_0}^{Q^2\alpha_{N_{I,M}(Q)}} f \left(\begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} W_{H,Q}(0) \right) ds, \quad (6.1)$$

noting that $h_{Q^2\alpha_i} W_{H,Q}(0) = W_{H,Q}(\alpha_i)$. We wish to show that the sequence $(\rho_{Q,I,M}^R)$ converges weakly to $\frac{|I|\mu_{G/H}}{\mu_{G/H}(G/H)}$. Notice that

$$\begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} W_{H,Q}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{Q^2} \\ 0 & 1 \end{pmatrix} H.$$

So (6.1) can be written as

$$\int f d\rho_{Q,I,M}^R = \int_{\alpha_0}^{\alpha_{N_{I,M}(Q)}} (\tilde{f} \circ g_Q) \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} H \right) dt,$$

where $\tilde{f} : G/H \rightarrow \mathbb{R}$ is the composition of left multiplication on G/H by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ followed by f , and $g_Q : G/H \rightarrow G/H$ is left multiplication by $\begin{pmatrix} Q & 0 \\ 0 & Q^{-1} \end{pmatrix}$. Since $\bigcup_{Q \in \mathbb{N}} \mathcal{F}_M(Q)$ is dense in $[0, 1]$, $\alpha_0 \rightarrow x_1$ and $\alpha_{N_{I,M}(Q)} \rightarrow x_2$ as $Q \rightarrow \infty$. So if we define the measure $\rho_{Q,I,M}^{R'}$ on G/H such that

$$\int f d\rho_{Q,I,M}^{R'} = \int_{x_1}^{x_2} (\tilde{f} \circ g_Q) \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} H \right) dt,$$

for all bounded, measurable functions $f : G/H \rightarrow \mathbb{R}$, then it is clear that $\rho_{Q,I,M}^R - \rho_{Q,I,M}^{R'} \rightarrow 0$ weakly. Thus to show that $\rho_{Q,I,M}^R \rightarrow \frac{|I|\mu_{G/H}}{\mu_{G/H}(G/H)}$ weakly, it suffices to prove that $\rho_{Q,I,M}^{R'} \rightarrow \frac{|I|\mu_{G/H}}{\mu_{G/H}(G/H)}$ weakly.

This convergence is mainly a consequence of the equidistribution of closed horocycles in G/H [19], [11]. The argument in [11] can be used to prove that

$$\lim_{Q \rightarrow \infty} \int_{x_1}^{x_2} (\tilde{f} \circ g_Q) \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} H \right) dt = \frac{x_2 - x_1}{\mu_{G/H}(G/H)} \int_{G/H} f d\mu_{G/H} \quad (6.2)$$

for all functions $f : G/H \rightarrow \mathbb{R}$ that are bounded and uniformly continuous.

We give this argument in detail. Let $f : G/H \rightarrow \mathbb{R}$ be bounded and uniformly continuous. For $\theta, t, y \in \mathbb{R}$ with $y > 0$, we make the following definitions:

$$k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad a_y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}, \quad u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

By the uniform continuity of f , for a given $\epsilon > 0$, there is a $\delta \in (0, 1)$ such that if $|\theta|, |y - 1| \leq \delta$, then

$$|(f \circ g_Q)(k_\theta a_y u_t H) - (f \circ g_Q)(u_t H)| < \epsilon$$

for all $Q \in \mathbb{N}$ and $t \in [0, 1]$. So if we let

$$B = \{k_\theta a_y u_t H : \theta \in [0, \delta], t \in [x_1, x_2], y \in [1 - \delta, 1]\}$$

and define $\overline{f \circ g_Q} : B \rightarrow \mathbb{R}$ by $\overline{f \circ g_Q}(k_\theta a_y u_t H) = f \circ g_Q(u_t H)$, then

$$\left| \int_B \overline{f \circ g_Q} d\mu_{G/H} - \int_B f \circ g_Q d\mu_{G/H} \right| \leq \epsilon \cdot \mu_{G/H}(B)$$

for all $Q \in \mathbb{N}$. Now notice that

$$\begin{aligned} \int_B \overline{f \circ g_Q} d\mu_{G/H} &= \int_0^\delta \int_{1-\delta}^1 \int_{x_1}^{x_2} \overline{f \circ g_Q}(k_\theta a_y u_t) dt dy d\theta \\ &= \frac{\mu_{G/H}(B)}{x_2 - x_1} \int_{x_1}^{x_2} (f \circ g_Q)(u_t H) dt, \end{aligned}$$

and so

$$\left| \int_{x_1}^{x_2} (f \circ g_Q)(u_t H) dt - \frac{x_2 - x_1}{\mu_{G/H}(B)} \int_B f \circ g_Q d\mu_{G/H} \right| \leq \epsilon(x_2 - x_1). \quad (6.3)$$

By the Howe-Moore theorem, the geodesic flow $\{g_s : s > 0\}$ is mixing on G/H , and so

$$\lim_{Q \rightarrow \infty} \int_B f \circ g_Q d\mu_{G/H} = \frac{\mu_{G/H}(B)}{\mu_{G/H}(G/H)} \int f d\mu_{G/H}.$$

Hence by (6.3), we have

$$\lim_{Q \rightarrow \infty} \int_{x_1}^{x_2} (f \circ g_Q)(u_t H) dt = \frac{x_2 - x_1}{\mu_{G/H}(G/H)} \int_{G/H} f d\mu_{G/H},$$

noting that $\epsilon > 0$ was chosen arbitrarily. This completes the proof of (6.2).

Next, noting that $\mu_{G/H}$ is left G -invariant, we have

$$\lim_{Q \rightarrow \infty} \int f d\rho_{Q,I,M}^{R'} = \frac{|I|}{\mu_{G/H}(G/H)} \int_{G/H} \tilde{f} d\mu_{G/H} = \frac{|I|}{\mu_{G/H}(G/H)} \int_{G/H} f d\mu_{G/H},$$

for every bounded, uniformly continuous function $f : G/H \rightarrow \mathbb{R}$. By the Portmanteau theorem, this is equivalent to saying that $\rho_{Q,I,M}^{R'} \rightarrow \frac{|I|\mu_{G/H}}{\mu_{G/H}(G/H)}$ weakly, which then implies that $\rho_{Q,I,M}^R \rightarrow \frac{|I|\mu_{G/H}}{\mu_{G/H}(G/H)}$ weakly.

Our next step is to prove that if $\pi_M : G/H \rightarrow \Omega_M$ is the projection $(x, s) \mapsto x$, where we are viewing G/H as $\{(x, s) \in \Omega_M \times \mathbb{R} : 0 \leq s < r_M(x)\}$, then $\pi_{M*} \rho_{Q,I,M}^R \rightarrow \frac{|I|\pi_{M*}\mu_{G/H}}{\mu_{G/H}(G/H)}$ weakly. So let $f \in C(\Omega_M)$ be nonnegative and bounded. For a given $\epsilon > 0$, let $g_\epsilon : \Omega_M \rightarrow \mathbb{R}$ be a continuous function such that $g_\epsilon \leq r_M$ and $\int_{\Omega_M} (r_M - g_\epsilon) d\mu_{\Omega''} < \frac{\epsilon}{2}$. Then

$$O_\epsilon = \{h_s p_{a,b} m_i H : (a, b) \in \text{int}(\mathcal{T}), m_i H \in M, 0 < s < g_\epsilon(p_{a,b} \gamma_i H)\}$$

is an open subset of G/H in which $\mu_{G/H}((G/H) \setminus O_\epsilon) < \frac{\epsilon}{2}$. So by the inner regularity of $\mu_{G/H}$ and Urysohn's lemma, there is a continuous function $\chi_\epsilon : G/H \rightarrow [0, 1]$ such that $\text{Supp } \chi_\epsilon \subseteq O_\epsilon$ and $\chi_\epsilon^{-1}(\{1\})$ is a compact subset of O_ϵ with $\mu_{G/H}((G/H) \setminus \chi_\epsilon^{-1}(\{1\})) < \epsilon$.

Now notice that π_M is continuous on O_ϵ , and therefore $f_{\epsilon,1} = \chi_\epsilon \cdot (f \circ \beta)$, $f_{\epsilon,2} = P - \chi_\epsilon \cdot (P - f \circ \pi_M) \in C(G/H)$, where $P > 0$ is a constant such that $f \leq P$. Thus

$$\lim_{Q \rightarrow \infty} \int_{G/H} f_{\epsilon,j} d\rho_{Q,I,M}^R = \frac{|I|}{\mu_{G/H}(G/H)} \int_{G/H} f_{\epsilon,j} d\mu_{G/H}, \quad j = 1, 2.$$

Since $f_{\epsilon,1} \leq f \circ \pi_M \leq f_{\epsilon,2}$, we also have

$$\begin{aligned} \liminf_{Q \rightarrow \infty} \int_{G/H} f \circ \pi_M d\rho_{Q,I,M}^R &\geq \frac{|I|}{\mu_{G/H}(G/H)} \int_{G/H} f_{\epsilon,1} d\mu_{G/H} \text{ and} \\ \limsup_{Q \rightarrow \infty} \int_{G/H} f \circ \pi_M d\rho_{Q,I,M}^R &\leq \frac{|I|}{\mu_{G/H}(G/H)} \int_{G/H} f_{\epsilon,2} d\mu_{G/H}. \end{aligned}$$

By the properties of χ_ϵ ,

$$\begin{aligned} \int_{G/H} f \circ \pi_M d\mu_{G/H} &\leq \int_{G/H} f_{\epsilon,1} d\mu_{G/H} + P\epsilon \text{ and} \\ \int_{G/H} f \circ \pi_M d\mu_{G/H} &\geq \int_{G/H} f_{\epsilon,2} d\mu_{G/H} - P\epsilon, \end{aligned}$$

and therefore

$$\begin{aligned} \liminf_{Q \rightarrow \infty} \int_{G/H} f \circ \pi_M d\rho_{Q,I,M}^R &\geq \frac{|I|}{\mu_{G/H}(G/H)} \left(\int_{G/H} f \circ \pi_M d\mu_{G/H} - P\epsilon \right) \text{ and} \\ \limsup_{Q \rightarrow \infty} \int_{G/H} f \circ \pi_M d\rho_{Q,I,M}^R &\leq \frac{|I|}{\mu_{G/H}(G/H)} \left(\int_{G/H} f \circ \pi_M d\mu_{G/H} + P\epsilon \right). \end{aligned}$$

Letting $\epsilon \rightarrow 0$ then yields

$$\lim_{Q \rightarrow \infty} \int_{G/H} f \circ \pi_M d\rho_{Q,I,M}^R = \frac{|I|}{\mu_{G/H}(G/H)} \int_{G/H} f \circ \pi_M d\mu_{G/H},$$

proving that $\pi_{M*}\rho_{Q,I,M}^R \rightarrow \frac{|I|\pi_{M*}\mu_{G/H}}{\mu_{G/H}(G/H)}$ weakly.

As noted in Section 5, $\frac{1}{r_M}$ can be well approximated in $L^1(\Omega_M, \mu_{\Omega''}|\Omega_M)$ from above and below by continuous functions $g_{2,\epsilon}$ and $g_{3,\epsilon}$, and so one can easily show that $\frac{1}{r_M}\pi_{M*}\rho_{Q,I,M}^R \rightarrow \frac{|I|\pi_{M*}\mu_{G/H}}{r_M\mu_{G/H}(G/H)}$ weakly using the fact that

$$g_{2,\epsilon}\pi_{M*}\rho_{Q,I,M}^R \rightarrow \frac{g_{2,\epsilon}|I|\pi_{M*}\mu_{G/H}}{\mu_{G/H}(G/H)} \quad \text{and} \quad g_{3,\epsilon}\pi_{M*}\rho_{Q,I,M}^R \rightarrow \frac{g_{3,\epsilon}|I|\pi_{M*}\mu_{G/H}}{\mu_{G/H}(G/H)}$$

weakly. Notice that

$$\frac{1}{r_M}\pi_{M*}\rho_{Q,I,M}^R = \frac{N_{I,M}(Q)}{Q^2}\rho_{Q,I,M} \quad \text{and} \quad \frac{|I|\pi_{M*}\mu_{G/H}}{r_M\mu_{G/H}(G/H)} = \frac{|I|\mu_{\Omega''}|\Omega_M}{\mu_{G/H}(G/H)},$$

and hence $\frac{N_{I,M}(Q)}{Q^2}\rho_{Q,I,M} \rightarrow \frac{|I|\mu_{\Omega''}|\Omega_M}{\mu_{G/H}(G/H)}$ weakly. Since $\rho_{Q,I,M}$ is a probability measure for all $Q \in \mathbb{N}$, we have

$$\lim_{Q \rightarrow \infty} \frac{N_{I,M}(Q)}{Q^2} = \lim_{Q \rightarrow \infty} \frac{N_{I,M}(Q)}{Q^2}\rho_{Q,I,M}(\Omega_M) = \frac{|I|\mu_{\Omega''}(\Omega_M)}{\mu_{G/H}(G/H)} = \frac{|I|(\#M)}{\mu_{G/H}(G/H)},$$

implying that $N_{I,M}(Q) \sim \frac{|I|(\#M)Q^2}{\mu_{G/H}(G/H)} = \frac{3|I|(\#M)Q^2}{\pi^2[\Gamma:H]}$. This proves the equidistribution of $(\mathcal{F}_M(Q))$ in $[0, 1]$, and the weak convergence

$$\rho_{Q,I,M} \rightarrow \frac{1}{\#M} \mu_{\Omega''}|_{\Omega_M},$$

completing the proof of Theorem 1.

7. THE REPULSION GAP FOR FAREY FRACTIONS $\frac{a}{q}$ SUCH THAT $q \equiv 1 \pmod{m}$

We conclude by determining the repulsion gap for Farey fractions with denominators congruent to 1 modulo m . For a given increasing sequence $\mathcal{A} := (A_n)$ of subsets of $[0, 1]$ with a limiting gap measure $\nu_{\mathcal{A}}$, we define the *repulsion gap* of \mathcal{A} to be

$$K_{\mathcal{A}} := \sup\{c \geq 0 : \nu_{\mathcal{A}}([0, c]) = 0\}.$$

This means that if $\Delta_{av}(A_n)$ is the average gap between consecutive elements in A_n , then for a given $\epsilon \in (0, K_{\mathcal{A}})$,

$$\lim_{n \rightarrow \infty} \frac{\#\{x, x' \text{ consecutive in } A_n : x' - x \leq \epsilon \Delta_{av}(A_n)\}}{\#A_n - 1} = 0.$$

In other words, the proportion of the number of gaps of elements in A_n that are smaller than $\epsilon \Delta_{av}(A_n)$ approaches 0 as $n \rightarrow \infty$. So $K_{\mathcal{A}}$ provides a measure for how big a large proportion of the gaps in A_n must be for large n .

Let $I \subseteq [0, 1]$ be a subinterval, $m \in \mathbb{N}$, and

$$A = \{(a, 1) \bmod m : a \in \{0, \dots, m-1\}\} \subseteq (\mathbb{Z}/m\mathbb{Z})^2$$

so that $\mathcal{F}_{I,m,A}(Q)$ is the set of fractions $\frac{a}{q} \in \mathcal{F}(Q) \cap I$ with $q \equiv 1 \pmod{m}$. We now compute the repulsion gap for the sequence $(\mathcal{F}_{I,m,A}(Q))$. First note that $\mathcal{F}_{I,m,A}(Q) = \mathcal{F}_{I,M}(Q)$, where M is the set of cosets of the form $\begin{pmatrix} a & b \\ -1 & d \end{pmatrix} \Gamma(m)$ in $\Gamma/\Gamma(m)$, where a, b , and d are any integers such that $ad + b = 1$. It is well known that $[\Gamma : \Gamma(m)] = m^3 \prod_{p|m} (1 - \frac{1}{p^2})$. Also, since the congruence $ad + b \equiv 1 \pmod{m}$ has m^2 solutions and each coset of $\Gamma/\Gamma(m)$ is completely determined by the congruence classes modulo m of the entries of one of its elements, there are m^2 cosets in M . So by (4.4), the repulsion gap of $(\mathcal{F}_{I,m,A}(Q))$ is

$$\frac{3c'}{\pi^2 m \prod_{p|m} (1 - \frac{1}{p^2})}$$

where $c' = \sup\{c \geq 0 : \mu_{\Omega''}(r_M^{-1}[0, c]) = 0\}$. We have previously found that for a given nonnegative fraction $\frac{c_1}{c_3}$, $f_{c_1/c_3}^{-1}[0, c]$ is nonempty only if $c \geq \max\{c_1, c_3\}$. Also, $f_0^{-1}[0, c]$ is a subset of

$$\{p_{a,b} m_i H : b \geq \frac{1}{ca}, i \in \{1, \dots, k\}\}$$

which clearly has positive $\mu_{\Omega''}$ -measure if and only if $c > 1$. Thus, we have $c' \geq 1$. On the other hand, notice that if $m_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $m_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $m_1 \Gamma(m), m_2 \Gamma(m) \in M$ and

$$m_1 m_2^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in m_1 \Gamma(m) m_2^{-1}.$$

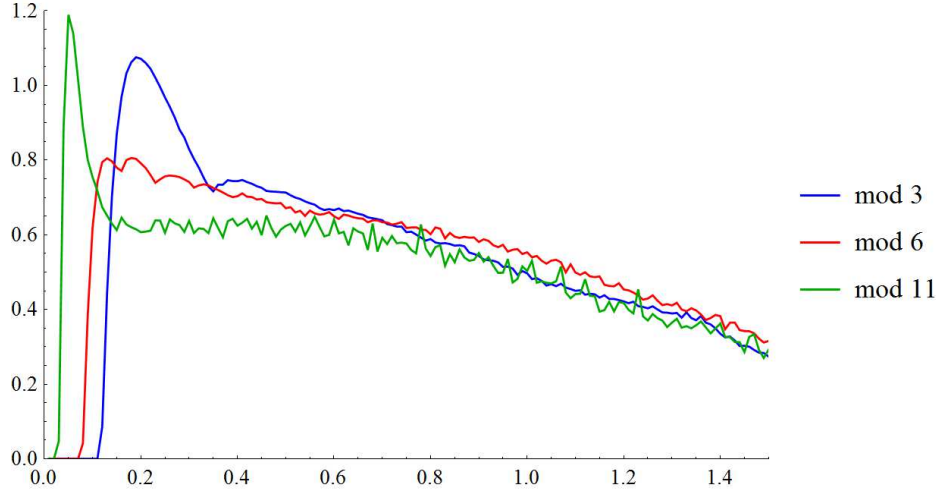


FIGURE 3. Limiting gap densities for fractions with denominators congruent to 1 modulo 3, 6, and 11

This implies that $f_0^{-1}[0, c]$ contains the set $\{p_{a,b}m_1H : b \geq \frac{1}{ca}\}$, which has positive $\mu_{\Omega''}$ -measure when $c > 1$. This proves that $c' = 1$, and hence the repulsion gap of $(\mathcal{F}_{I,m,A}(Q))$ is

$$\frac{3}{\pi^2 m \prod_{p|m} (1 - \frac{1}{p^2})}.$$

This can be seen in the numerical approximations of the densities of limiting gap measures for fractions congruent to 1 modulo 3, 6, and 11, in Figure 3.

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